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# The Forcing Edge-to-vertex Geodetic Number and the Forcing Edge covering Number of a Graph

**S. Sujitha**

Department of Mathematics

Holy Cross College (Autonomous), Nagercoil-629 004.

 Email: sujivenkit@gmail.com

 **Abstract**

In this paper we extend the study of the edge-to-vertex geodetic number, the edge covering number and their corresponding forcing number of a graph. The goal of this paper is to find the relation between the above four. The forcing edge-to-vertex geodetic number was introduced and studied in [4] and the forcing edge covering number of a graph was studied in [2]. Also, we prove for every integers *a*, *b*, *c* and *d* with 0 ≤ *a* ≤ *b* < *c* < *d*, and c> *b* + 1, *d* > *b* + *c* – *a*, there exists a connected graph *G* such that *fev*(*G*) = *a*, *fß′* (*G*) = *b*, *gev*(*G*) = *c* and *ß′* (*G*) = *d*.

**Keywords:**edge-to-vertex geodetic number,forcing edge-to-vertex geodetic number, edge covering number, forcing edge-covering number.

**AMS Subject Classification:** 05C12.

 **1.Introduction**

By a graph *G* = (*V*, *E*), we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of *G* are denoted by *p* and *q* respectively. For basic definitions and terminologies we refer to [1]. An edge covering of *G* is a subset *L* ⊆ *E*(*G*) such that each vertex of *G* is end of some edge in *L*. The number of edges in a minimum edge covering of *G*, denoted by *ß′* (*G*) is the edge covering number of *G*. A subset *T*  *L* is called a forcing subset for *L* if *L* is the unique minimum edge covering containing *T*. A forcing subset for *L* of minimum cardinality is a minimum forcing subsetof *L*. The forcing edge covering number of *L*, denoted by *fß′* (*L*), is the cardinality of a minimum forcing subset of *L*. The forcing edge covering number of *G*, denoted by *fß′* (*G*), is *fß′* (*G*) = min {*fß′* (*L*)}, where the minimum is taken over all minimum edge coverings *L* in *G*.

 A set *S* ⊆ *E*(*G*) is called an *edge-to-vertex geodetic set* of *G* if every vertex of *G* is either incident with an edge of *S* or lies on a geodesic joining a pair of edges of *S*. The minimum cardinality of an edge-to-vertexgeodetic set of *G* is *gev*(*G*). Any edge-to-vertex geodetic set of cardinality *gev*(*G*) is called an *edge-to-vertex geodetic basis* of *G*. A subset *T*  *S* is called a *forcing subset* for *S* if *S* is the unique minimum edge-to-vertex geodetic set containing *T*. A forcing subset for *S* of minimum cardinality is a *minimum forcing subset* of *S*. The *forcing edge-to-vertex geodetic number* of *S*, denoted by *fev*(*S*), is the cardinality of a minimum forcing subset of *S*. The forcing edge-to-vertex geodetic number of *G*, denoted by *fev*(*G*), is *fev*(*G*) = min{*fev*(*S*)}, where the minimum is taken over all minimum edge-to-vertex geodetic sets *S* in *G*. In this paper we study the relation between the forcing concept of the minimum edge-to-vertex geodetic set and the minimum edge covering of a connected graph. Throughout the following *G* denotes a connected graph with at least three vertices. The following Theorems are used in the sequel.

**Theorem 1.1 [2]** Every end-edge of a connected graph *G* belongs to every edge covering of *G*.

**Theorem 1.2 [2]** Let *G* be a connected graph with size *q*. Then *ß′* (*G*) = *q* if and only if

 *G* is a star.

**Theorem 1.3 [2]** For the star *G* = *K*1,*q* ( *q* ≥ 2), *fß′* (*G*)= 0.

**Theorem 1.4 [2]** Let *G* be a connected graph and *W* be the set of all edge covering

 edges of *G*. Then *fß′* (*G*) ≤ *ß′* (*G*) – | *W* |.

**Theorem 1.5 [2]** Let *G* be a connected graph. Then

1. *fß′* (*G*) = 0 if and only if *G* has a unique minimum edge covering.
2. *fß′* (*G*) = 1 if and only if *G* has at least two minimum edge coverings, one of which is a unique minimum edge covering containing one of its elements, and
3. *fß′* (*G*) =  *ß′* (*G*) if and only if no minimum edge covering of *G* is the unique minimum edge covering containing any of its proper subsets.

**Theorem 1.6 [4]** For any connected graph *G* ofsize *q* ≥ 2, *gev*(*G*) = *q* if and only if *G* is a star.

**Theorem 1.7 [4]** Every end-edge of a connected graph *G* belongs to every edge-to-vertex geodetic set of *G*.

**Theorem 1.8 [4]** For a non-trivial tree *G* = *T* of size *q* ≥ 2, *fev*(*G*) = 0.

**Theorem1.9 [4]** Let *G* be a connected graph and *W* be the set of all edge-to-vertex geodetic edges of *G*.

 Then *fev*(*G*) ≤ *gev*(*G*) – | *W* |.

**Theorem 1.10 [4]** Let *G* be a connected graph. Then

1. *fev*(*G*) = 0 if and only if *G* has a unique minimum edge-to-vertex geodetic set.
2. *fev*(*G*) = 1 if and only if *G* has at least two minimum edge-to-vertex geodetic sets, one of which is a unique minimum edge-to-vertex geodetic set containing one of its elements, and
3. *fev*(*G*) =  *gev*(*G*) if and only if no minimum edge-to-vertex geodetic set of *G* is the unique minimum edge-to-vertex geodetic set containing any of its proper subsets.

**Theorem** **2.1.** For every pair *a*, *b* of integers with 2 ≤ *a* ≤ *b*, there exists a connected graph *G* such that *fev*(*G*) = *fß′* (*G*) = 0; and *gev*(*G*) = *a* and *ß′* (*G*) = *b*.

**Proof.** If *a* = *b*, let *G=K*1,*a*. Then by Theorems 1.2 and 1.6, *gev*(*G*) = *a* =  *ß′* (*G*) and by Theorems 1.8 and 1.3, *fev*(*G*) *= fß′* (*G*) *=* 0. For *a* < *b*, let *P*: *x*1, *x*2, *x*3, *x*4 be a path of length 3. For each integer *i* with1 ≤ *i* ≤ *b – a –*1, let *Fi* : *ui* ,*vi*be the path of order 2. Let *G* be the graph obtained from the graphs *P* and *Fi* (1 ≤ *i* ≤ *b – a –*1) by adding the 2(*b – a –*1) edges *x*1*ui* and *x*4*vi* for 1 ≤ *i* ≤ *b – a –*1 and also adding the end-edges *x*1*z* and *x*4*zj*(1 ≤ *j* ≤ *a –*1). The graph *G* is shown in Figure 2.1.

Let *Z* = {*x*1*z*, *x*4*z*1, *x*4*z*2, …, *x*4*za*-1} be the set of all end-edges of *G*. Then by Theorem 1.7, *Z* is a subset of every edge-to-vertex geodetic set of *G*. But it is clear that *Z* is an edge-to-vertex geodetic set of *G*. Since *Z* is the unique minimum edge-to-vertex geodetic set of *G* we have *gev*(*G*) = *a* and hence by Theorem 1.10(a) *fev*(*G*) = 0. Since the edges *uivi*(1≤ *i* ≤ *b* – *a* –1) do not lie on *Z*, we see that *Z* is not an edge covering set of *G*. Now it is easily seen that *W* = *Z* ∪ {*x*2*x*3, *u*1*v*1, *u*2*v*2, …, *ub*-*a*-1*vb*-*a*-1} is the unique minimum edge covering set of *G* such that *ß′* (*G*) = | *W* | = *b* and hence by Theorem 1.5(*a*), *fß′* (*G*) = 0. **∎**

*G*

Figure 2.1

*za-*1

*z*2

*z*1

*x*3

*x*2

*x*1

*z*

*x*4

*ub-a-*1

*u*2

*u*1

*v*1

*v*2

*vb-a-*1

**Theorem** **2.2.** For every integers *a*, *b* and *c* with 0 ≤ *a* < *b* < *c* and *b*> *a* + 1, *c* = *a +b* there exists a connected graph *G* such that *fev*(*G*) = 0 ,  *fß′* (*G*) = *a*, *gev*(*G*) = *b* and *ß′*(*G*) = *c*.

**Proof.** We consider two cases.

**Case 1.** *a* = 0. Then the graph *G* constructed in Theorem 2.1 satisfies the requirement of the theorem.

**Case 2.** *a* ≥ 1. Let *P*4 : *x*, *y*, *v*1, *z* be a path of order 4. Now add *b* – 1 new vertices *z*1, *z*2, …, *zb*-1 to *P*4 and join each to *z*, there by producing a tree *T*. Then add *a* – 1 new verticesv2, *v*3,…, *va*–1, *va*to *T* and join each to both *y* and *z*, and obtaining the graph *G* of figure 2.2.

 Let *Z* = {*xy*, *zz*1, *zz*2, …, *zzb-*1} be the set of all end-edges of *G*. By Theorem 1.7, *Z* is a subset of every edge-to-vertex geodetic set of *G*. It is easily verified that *Z* is the unique minimum edge-to-vertex geodetic set of *G* and so *gev*(*G*) = *b* and hence by Theorem 1.10(a) *fev*(*G*) = 0. Next we show that *ß′* (*G*) = *c*. Let *S* be any edge covering set of *G*. Then by Theorem 1.1, *Z* *S*. It is clear that *Z* is not an edge covering set of *G*. Let *Hi* = {*hi*, *ki*}, where *hi* = *yvi* and *ki* = *zvi* (1 ≤ *i* ≤ *a*). We observe that every edge covering of *G* must contain at least one vertex from each *Hi* (1 ≤ *i* ≤ *a*) so that *ß′* (*G*) ≥ *b + a* = *c*. Now, *S*1 = *Z* ∪ {*h*1, *h*2, …, *ha*} is an edge covering set of *G* so that *ß′* (*G*) ≤ *b + a* = *c*. Thus *ß′* (*G*) = *c*. Since every edge covering contains *Z,* it follows from Theorem 1.4, *fß′* (*G*) ≤ *ß′* (*G*) – | *Z* | = *c* – *b* = *a*. Now, since *ß′* (*G*) = *c* and every edge covering of *G* contains *Z*, it is easily seen that every edge covering *S* is of the form *Z* ∪ {*d*1, *d*2, …, *da*} ,where *di**Hi* (1 ≤ *i* ≤ *a*). Let *T* be any proper subset of *S* with | *T* | < *a*. Then it is clear that there exists some *j* such that *T* ∩ *Hj* = Φ, which shows that *fß′* (*G*) = *a*.

 **∎**

*x*

*v*2

*v*3

*va*

 *y*

*G*

Figure 2.2

*zb-*1

*z*2

*z*1

*z*

*v*1

**Theorem** **2.3.** For every integers *a*, *b* with 0 ≤ *a* < *b* and *b* > *a* + 1, there exists a connected graph *G* such that *fev*(*G*) = *fß′* (*G*) = *a*, *gev*(*G*) = *ß′* (*G*) = *b*.

**Proof.** We consider two cases.

**Case 1.** *a* = 0. Let *G=K*1*,b.* Then by Theorems 1.6 and 1.2, *gev*(*G*) = *b* = *ß′* (*G*) and by Theorems 1.8 and 1.3, *fev*(*G*) *= fß′* (*G*) *=* 0.

**Case 2.** *a* ≥ 1. Let *P*4 : *x*, *y*, *v*1, *z* be a path of order 4. Now add *b* – *a* – 1 new vertices *z*1, *z*2, …, *zb*-*a*-1 to *P*4 and join each to *z*, there by producing a tree *T*. Let *G* be the graph obtained from *T* by adding *a* – 1 new vertices *v*2, *v*3, …, *va*-1, *va* and join each to both *y* and *z*. Also, join the vertices *y* and *z*, and is shown in figure 2.3.

 Let *Z* = {*xy*, *zz*1, *zz*2, …, *zzb-a-*1} be the set of all end- edges of *G*. Let *S* be any edge-to-vertex geodetic set of *G.* Then by Theorem 1.7, *Z**S*. First we show that *gev*(*G*) = *b*. Clearly *Z* is not an edge-to-vertex geodetic set of *G.* Let *Hi* = {*hi*, *ki*}, where *hi* = *yvi* and *ki* = *zvi* (1 ≤ *i* ≤ *a*). We observe that every edge-to-vertex geodetic set of *G* must contain at least one vertex from each *Hi* (1 ≤ *i* ≤ *a*). Thus *gev*(*G*) ≥ *b* – *a* + *a* = *b*. On the other hand since the set *S* = *Z* ∪ {*k*1, *k*2, *k*3,…, *ka*} is an edge-to-vertex geodetic set of *G*, it follows that *gev*(*G*) ≤ | *S* | = *b*. Hence *gev*(*G*) = *b*. Next we show that *fev*(*G*) = *a.* By Theorem 1.7, every edge-to-vertex geodetic set of *G* contains *Z* and so it follows from Theorem 1.9, *fev*(*G*) ≤ *gev*(*G*) –  *|Z|* = *a*. Now, since *gev*(*G*) = *b* and every *gev*-set of *G* contains *Z*, it is easily seen that every *gev*-set *S* is of the form *Z* ∪ {*c*1, *c*2, *c*3, … *ca*}, where *ci*  *Hi* (1 ≤ *i* ≤ *a*). Let *T* be any proper subset of *S* with |*T*| < *a*. Then it is clear that there exists some *j* such that *T* ∩ *Hj* = Φ, which shows that *fev*(*G*) = *a*. By the similar way we can prove *ß′* (*G*) = *b* and *fß′* (*G*) = *a*. **∎**

*x*

*v*2

*v*3

*va*

 *y*

*G*

Figure 2.3

*zb-a-*1

*z*2

*z*1

*z*

*v*1

**Theorem** **2.4.** For every integers *a*, *b*, *c* and *d* with 0 ≤ *a* ≤ *b* < *c* < *d*, and c> *b* + 1, *d* > *b* + *c* – *a*, there exists a connected graph *G* such that *fev*(*G*) = *a*, *fß′* (*G*) = *b*, *gev*(*G*) = *c* and *ß′* (*G*) = *d*.

**Proof.** We consider three cases.

**Case 1.** 0 = *a* = *b*. Then the graph *G* constructed in Theorem 2.1 satisfies the requirement of this theorem.

**Case 2.** 0 = *a* < *b*. Then the graph *G* constructed in Theorem 2.2 satisfies the requirement of this theorem.

**Case 3.**  0 < *a* < *b* < *c* < *d*, *a ≥* 1. Let *Pd* : *x*1, *x*2, *x*3, *x*4, *x*5, *x*6, *x*7, *x*8be a path of order 8. Now add *c* – *a* – 1 new vertices *z*1, *z*2, …, *zc*-*a*-1 to *Pd* and join *z*1 to *x*3, *z*2 to *x*5 and *z*3, *z*4, …, *zc*-*a*-1 to *x*8, there by producing a tree *T*. Then add new vertices *y*1, *y*2, …, *ya* –1, *ya* to *T* and join each to both *x*2 and *x*3. Also add *b* – *a* – 1 new vertices *n*1, *n*2, …, *nb*-*a*-1 to *T* and join each to both *x*3 and *x*5. For each integer *i* with1 ≤ *i* ≤ *d* – (*b+ c* – *a* + 1), let *Fi* : *ui*,*vi*be the path of order 2. Let *G* be the graph obtained from the graphs *T* and *Fi* (1 ≤ *i* ≤ *d* – (*b+ c* – *a +* 1)) by adding the 2 (*d* – (*b+ c* – *a +*1)) edges *x*5*uj* and *x*8*vj* for 1 ≤ *j* ≤ *d* – (*b+ c* – *a +*1). The graph *G* is shown in Figure 2.4.

 Let *Z* = {*x*1*x*2, *x*3*z*1, *x*5*z*2, *x*8*z*3, *x*8*z*4, …, *x*8*zc-a-*1} be the set of all end-edges of *G*. Let *S* be any edge-to-vertex geodetic set of *G.* Then by Theorem 1.7, *Z* *S*. First we show that *gev*(*G*) = *c*. Clearly *Z* is not an edge-to-vertex geodetic set of *G.* Let *Hi* = {*hi, ki*}, where *hi* = *x*2*yi* and *ki* = *x*3*yi* (1 ≤ *i* ≤ *a*). We observe that every edge-to-vertex geodetic set of *G* must contain at least one vertex from each *Hi* (1 ≤ *i* ≤ *a*). Thus *gev*(*G*) ≥ *c* – *a* + *a* = *c*. On the other hand, any edge-to-vertex geodetic set is of the form *S* = *Z* ∪ {*c*1, *c*2, *c*3, …, *ca*}, where *ci*  *Hi* (1≤ *i* ≤ *a*). Then as in earlier theorems it can be seen that *fev*(*G*) = *a* and *gev*(*G*) = *c*. Let *Qi*= {*ri*, *si*}, where *ri* = *x*3*ni* and *si* = *x*5*ni* (1 ≤ *i* ≤ *b* – *a* –1). It is clear that any edge covering set is of the form *W* = *Z* ∪ {*u*1*v*1, *u*2*v*2, …, *u d*- (*b+ c – a+*1) *vd*- (*b+ c – a+*1)} ∪ {*c*1, *c*2, *c*3, …, *ca*} ∪ {*d*1, *d*2, *d*3, …, *db*-*a-*1}, where *ci**Hi* (1 ≤ *i* ≤ *a*) and *dj* *Qi* (1 ≤ *j* ≤ *b* – *a* – 1). Then as in earlier theorems it can be seen that *fß′* (*G*) = *b* and *ß′* (*G*) = *d*. **∎**

*G*

Figure 2.4

*zc-a-*1

*z*4

*z*3

*x*7

*x*6

*x*5

*z*2

*x*8

*ud-*(*b+c-a+*1)

*u*2

*u*1

*v*1

*v*2

*z*1

*n*1

*n*2

*nb-a-*1

 *x*3

*x*4

*y*1

*y*2

*ya*

 *x*2

*vd-*(*b+c-a+*1)

 *x*1

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